Homoclinic tangencies to resonant saddles and discrete Lorenz attractors

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Abstract. We study bifurcations of periodic orbits in three parameter general unfoldings of certain types quadratic homoclinic tangencies to saddle fixed points. We apply the rescaling technique to first return (Poincaré) maps and show that the rescaled maps can be brought to a map asymptotically close to the 3D Henon map $\bar{x} = y, \bar{y} = z, \bar{z} = M_1 + M_2 y + Bx - z^2$ which, as known [1], exhibits wild hyperbolic Lorenz-like attractors in some open domains of the parameters. Based on this, we prove the existence of infinite cascades of Lorenz-like attractors[1].

Key words: Homoclinic tangency, rescaling, 3D Hénon map, bifurcation.
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1   Introduction

In [1] it was discovered that the three-dimensional Henon map

$$\bar{x} = y, \ \bar{y} = z, \ \bar{z} = M_1 + M_2 y + Bx - z^2, \tag{1.1}$$

where (M_1, M_2, B) are parameters (B is the Jacobian of map), can possess strange attractors that seem very similar to the Lorenz attractors, see fig. 1. Later it was shown that such *discrete Lorenz attractors* can arise as result of simple, universal and natural bifurcation scenarios realizing in one-parameter families of three-dimensional maps [44, 51]. This means, in fact, that the discrete Lorenz attractors can be met widely in applications. For instance, in [52, 53] such attractors were found in nonholonomic models of rattleback (called also as a Celtic stone). See also [41, 53] where various types of strange homoclinic attractors, including discrete Lorenz ones, were investigated.

The similarity between the discrete and classical Lorenz attractors appears to be not accidental and it can be explained by various reasons. Thus, it is well known that the classical Lorenz attractor can be born as a result of local bifurcations of an equilibrium state with three zero eigenvalues when a flow possesses a (Lorenzian) symmetry [45].
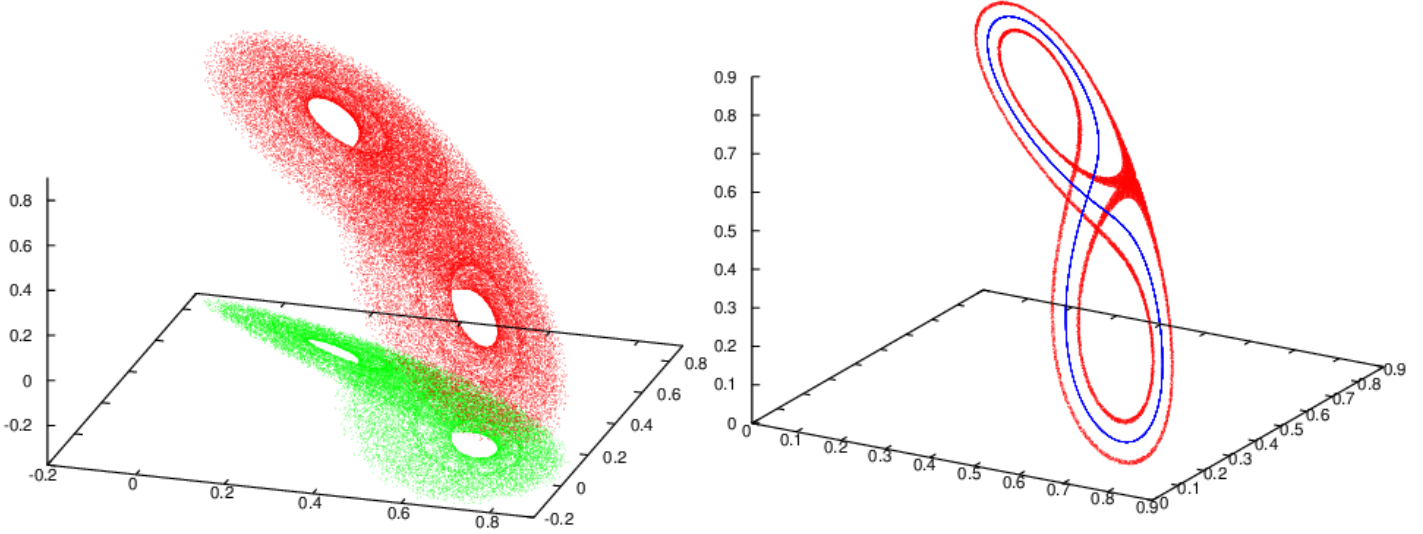

Figure 1: Plots of attractors of map (1.1) observed numerically in [1] for $M_1 = 0, B = 0.7$ and $M_2 = 0.85$ (left) or $M_2 = 0.815$ (right). In the left panel, the projection on the (x, y)-plane is also displayed. In the right panel, a "figure-eight" saddle closed invariant curve inside the lacuna is shown. Note the similarity to the Lorenz attractors of the Shimizu-Morioka system, see [54, 55, 56].

Analogously for maps, discrete Lorenz attractors can arise under bifurcations of fixed points with multipliers $(-1, -1, +1)$, in this case the required local symmetry exists automatically due to negative multipliers. As it was shown in [1, 41], the second iteration of the map near this point can be embedded into a flow up to asymptotically small periodic non-autonomous terms. The corresponding flow normal form of such bifurcations coincides with the well-known Shimizu-Morioka model, which, in turn, exhibits the Lorenz attractor for certain parameter values [39, 50]. Thus, we can consider the attractor in the map as the one of the Poincaré map (period map) of a periodically perturbed system with the Lorenz attractor. On the other hand, as it was shown in paper [42] by Turaev and Shilnikov, such discrete attractor is genuine in the sense that every its orbit has positive maximal Lyapunov exponent[2].

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[2]Moreover, it is a wild pseudohyperbolic attractor [42], since it allows homoclinic tangencies (i.e. contains Newhouse wild hyperbolic sets [49]) and has an adsorbing domain inside which the differential DT, for the map itself and all close maps can be decomposed into a direct sum of transverse invariant subspaces $\mathcal{W}^{ss}$ and $\mathcal{W}^{eu}$ where $DT_{\mathcal{W}^{ss}}$ is strongly contacting and $DT_{\mathcal{W}^{eu}}$ expands exponentially volume,

In the present paper we study bifurcations of three-dimensional diffeomorphisms with homoclinic tangencies, leading to the birth of descrete Lorenz attractors. Problems of this kind were previously analyzed in [43, 46, 47, 48, 4].

In [46, 47, 48] the birth of discrete Lorenz attractors from nontransversal heteroclinic cycles of three-dimensional diffeomorphisms was studied. Such a cycle contains two fixed points O_1 and O_2 of type $(2,1)$, i.e. with $\dim W^s(O_i) = 2, \dim W^u(O_i) = 1$, and one pair of stable and unstable manifolds intersect transversely and another pair has a quadratic tangency. It was assumed that at least one of the points O_1 and O_2 is a saddle-focus, see Fig. Moreover, in all cases the additional condition that the Jacobians of the map in points O_1 and O_2 are greater and less than one respectively was imposed (the so-called case of contracting-expanding maps). The birth of discrete Lorenz attractors was proved for three-parameter general unfoldings.

Remark. Naturally, three parameters are needed to allow generically the existence of triply degenerate fixed points in the corresponding first return maps. In such families the first return map can be rescaled to the form asymptotically close to map (1.1). Thus, using the results of [1] (see also [41] for more generic statement), we deduce the birth of discrete Lorenz attractors in close systems.

In the case of homoclinic tangencies to the saddle fixed point O of a three-dimensional diffeomorphism T the birth of Lorenz attractors was proved in the cases when:

1) [43], the point O is a saddle-focus with the unit Jacobian (saddle-focus of conservative type).

2) [4], the fixed point is a saddle with the unit Jacobian and the quadratic tangency is non-simple[3].

We note that the condition on Jacobians in all these cases is necessary for the existence of a non-trivial (three-dimensional) dynamics in the neighborhood of the homoclinic orbit [14]. Otherwise, if, for example, one has $J < 1$, all three-dimensional volumes will be contracted under the iterations of map T (near point O) and, hence, the dynamics of first return maps T^k for large k will be effectively two-dimensional, or even one-dimensional. Recall that, by definition [14], the *effective dimension* d_e of a bifurcation problem equals n, if periodic orbits with n multipliers equal ± 1 can appear at bifurcations but no orbits exist with more than n unit multipliers.

Formally, if to consider three-dimensional diffeomorphisms with homoclinic tangencies

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i.e. $\|DT^k_{\mathcal{W}^{ss}}\| < L\sigma^k$ and $|\det DT^k_{\mathcal{W}^{eu}}| > L\nu^k$ for some constants $L > 0, 0 < \sigma < 1 < \nu$ and all positive k.

[3]The definition of simple homoclinic tangency can be found in [12]. In particular, it assumes that the so-called extended unstable invariant manifold intersects transversely the leaf of the strong stable foliation in the point of tangency. The main cases of non-simple homoclinic tangencies in three-dimensional diffeomorphisms were considered in [2], see also condition **D** in §2 of the present paper.

3

to a hyperbolic saddle fixed point with $|J| = 1$, then d_e can be equal to 3 only in the following cases: (i) the point is a saddle-focus; (ii) the point is the saddle (all multipliers are real) and the tangency is not simple, and (iii) the point is a resonant saddle, i.e. it has two multipliers equal in the absolute value. Otherwise, the effective dimension is less than three since the direction of strong contraction is present [12, 3] for all nearby systems.

Cases (i) and (ii) were considered in [43] and [4] respectively. In this paper we consider the new case (iii) when the saddle is resonant. Note that if the resonance $\lambda_1 = \lambda_2$ takes place, we may perturb the map in such a way that the resulting map will have a saddle-focus fixed point with $|J| = 1$ and, hence, we can apply results of [43] to prove the birth of discrete Lorenz attractors. It not the case for the resonance $\lambda_1 = -\lambda_2$ which is of independent interest.

We consider the case when a fixed point O has multipliers $\lambda, -\lambda, \gamma$ such that $0 < \lambda < 1$, $|\gamma| > 1$ and $|\lambda^2 \gamma| = 1$. This means that O is a resonant saddle point of conservative type. Obviously, the bifurcation codimension of this problem is at least three and, as we will show, $d_e = 3$ in this case.

We show that in the three-parametric families f_μ, $\mu = (\mu_1, \mu_2, \mu_3)$ unfolding generally this type of a homoclinic tangency, in the parameter space there exist domains $\triangle_k \to \{\mu = 0\}$ as $k \to \infty$ such that for $\mu \in \triangle_k$ the first return map T_k possesses the discrete Lorenz attractor. Recall that the map T_k is constructed by the iterations of map f_μ, i.e. $T_k = f_\mu^k$, but the domain of its definition is a small box σ_0^k near some homoclinic point.

The paper consists of two paragraphs. In §2 we formulate our main result – Theorem 1 and construct the first return map of some small neighborhood of the homoclinic orbit. In §3 we prove Theorem 1.

2 Statement of the problem and formulation of main results.

We study bifurcations of three-dimensional diffeomorphisms of a special type (codimension two) quadratic homoclinic tangency to a saddle fixed point with the unit Jacobian. Namely, we assume that the initial diffeomorphism $f_0 \in C^r, r \geq 5$, satisfies the following conditions:

A) f has a saddle fixed point O with real multipliers $\lambda_1, \lambda_2, \gamma$ such that $0 < |\lambda_{1,2}| < 1 < |\gamma|$ and
$$J_0 \equiv |\lambda_1 \lambda_2 \gamma| = 1.$$

B) The stable $W^s(O)$ and unstable $W^u(O)$ invariant manifolds of O have a quadratic tangency at the points of some homoclinic orbit Γ_0.

C) The saddle O is resonant in the sense that $\lambda_1 = -\lambda_2 = \lambda > 0$.

Condition **A** means that the point O is a saddle of conservative type and dim $W^s(O) = 2$ and dim $W^u(O) = 1$. Condition **C** is an additional degeneracy of the saddle fixed point. We will consider smooth parameter families f_ε of diffeomorphisms (general unfoldings of conditions **A–C**), such that f_0 belongs to it for $\varepsilon = 0$.

Let $U \equiv U(O \cup \Gamma_0)$ be a sufficiently small fixed neighbourhood of Γ_0 that is a union of a neighbourhood U_0 of O and a number of neighbourhoods of those points of Γ_0 which lie outside U_0. Denote by T_0 the restriction of the diffeomorphism f_ε onto U_0. We call T_0 *a local map.* By a linear transformation of coordinates in U_0, map T_0 can be written as

$$(\bar{x}_1, \bar{x}_2, \bar{y}) = (\lambda_1 x_1, \lambda_2 x_2, \gamma y) \ + \ h.o.t.$$

The origin $O = (0,0,0)$ is a fixed point of T_0, the stable manifold $W^s(O)$ is tangent at O to the (x_1, x_2)-plane and the unstable manifold $W^u(O)$ is tangent at O to the y-axis. The intersection points of Γ_0 with U_0 belong to the set $W^s \cap W^u$ and accumulate to O at both forward and backward iterations. Thus, infinitely many points of Γ_0 lie on W^s_{loc} and W^u_{loc}. Let $M^+ \in W^s_{loc}$ and $M^- \in W^u_{loc}$ be two such points and let $M^+ = f_0^{n_0}(M^-)$ for some positive integer n_0. Let $\Pi^+ \subset U_0$ and $\Pi^- \subset U_0$ be small neighbourhoods of points M^+ and M^- respectively. The map $T_1 \equiv f_\varepsilon^{n_0} : \Pi^- \to \Pi^+$ is called *a global map.*

From [32, 27, 28, 22] it is known that there exists a C^r-change of coordinates (which is C^{r-2}-smooth in the parameters) bringing T_0 to the so-called *main normal form*:

$$
\begin{aligned}
\bar{x}_1 &= \lambda_1(\varepsilon)x_1 + O(\|x\|^2|y|) \ , \\
\bar{x}_2 &= \lambda_2(\varepsilon)x_2 + O(\|x\|^2|y|) \ , \\
\bar{y} &= \gamma(\varepsilon)y + O(\|x\||y|^2) \ .
\end{aligned}
\tag{2.1}
$$

The main peculiarity of this form is that in coordinates (2.1) the stable and unstable manifolds of the saddle fixed point are locally straightened, their equations are W^s : $\{y = 0\}$, W^u : $\{x_1 = 0, x_2 = 0\}$. The main normal form also allows to obtain a quite simple representation of the iterations of T_0. The latter can be formulated as the following lemma:

Lemma 1 [3] *For any positive integer k and for any sufficiently small ε the map $T_0^k(\varepsilon)$: $(x_0, y_0) \to (x_k, y_k)$ can be written in the following cross-form*

$$
\begin{aligned}
x_{k1} - \lambda_1^k(\varepsilon)x_{01} &= \hat{\lambda}^k \xi_{k1}(x_0, y_k, \varepsilon) \ , \\
x_{k2} - \lambda_2^k(\varepsilon)x_{02} &= \hat{\lambda}^k \xi_{k2}(x_0, y_k, \varepsilon) \ , \\
y_0 - \gamma(\varepsilon)^{-k}y_k &= \hat{\gamma}^{-k} \eta_k(x_0, y_k, \varepsilon) \ ,
\end{aligned}
\tag{2.2}
$$

where $\hat{\lambda}$ and $\hat{\gamma}$ are some constants such that $\hat{\lambda} = \lambda + \delta$, $\hat{\gamma} = \lambda|\gamma^{-1}| - \delta$ for some small $\delta > 0$ and functions ξ_k and η_k are uniformly bounded along with all derivatives up to order $(r - 2)$.

Next we construct the most appropriate form for the global map T_1 for all small ε. Let the chosen homoclinic points have coordinates $M^+ = M^+(x_1^+, x_2^+, 0) \in W^s_{loc}$ and $M^- = M^-(0, 0, y^-) \in W^u_{loc}$, where $(x_1^+)^2 + (x_2^+)^2 \neq 0$ and $y^- > 0$. At $\varepsilon = 0$ we have that $T_1 M^- = M^+$ and $T_1(W^u_{loc})$ and W^s_{loc} are tangent quadratically at the point M^+. Thus, the global map T_1 at $\varepsilon = 0$ can be written as the Taylor expansion near the point $(x_1 = 0, x_2 = 0, y = y^-)$:

$$
\begin{aligned}
\bar{x}_1 - x_1^+ &= a_{11}x_1 + a_{12}x_2 + b_1(y - y^-) + O(\|x\|^2) + O(\|x\|\|y - y^-\|) + O((y - y^-)^2) \\
\bar{x}_2 - x_2^+ &= a_{21}x_1 + a_{22}x_2 + b_2(y - y^-) + O(\|x\|^2) + O(\|x\|\|y - y^-\|) + O((y - y^-)^2) \\
\bar{y} &= c_1x_1 + c_2x_2 + d(y - y^-)^2 + O(\|x\|^2) + O(\|x\|\|y - y^-\|) + O(|y - y^-|^3)
\end{aligned}
$$

$$(2.3)$$

The equation of curve $T_1(W^u_{loc})$ at $\varepsilon = 0$ looks as follows (we put $x_1 = x_2 = 0$ in (2.3)):

$$
\begin{aligned}
\bar{x}_1 - x_1^+ &= b_1(y - y^-) + O((y - y^-)^2) \\
\bar{x}_2 - x_2^+ &= b_2(y - y^-) + O((y - y^-)^2) \\
\bar{y} &= d(y - y^-)^2 + O(|y - y^-|^3)
\end{aligned}
$$

$$(2.4)$$

This is a parametric equation (with parameter $(y - y^-)$) of the curve $T_1(W^u_{loc})$ in a neighbourhood of M^+. The equation of W^s_{loc} is $y = 0$. Since the initial homoclinic tangency is quadratic, it follows that $d \neq 0$, $b_1^2 + b_2^2 \neq 0$. Moreover, map $T_1(0)$ is a diffeomorphism, therefore

$$
J_1 = \det \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ c_1 & c_2 & 0 \end{pmatrix} \neq 0
$$

$$(2.5)$$

and, hence, $c_1^2 + c_2^2 \neq 0$.

At small ε the global map $T_1(\varepsilon)$ can be written in the following form (the Taylor expansion near the point $(x_1, x_2, y) = (0, 0, y^-(\varepsilon))$

$$
\begin{aligned}
\bar{x}_1 - x_1^+(\varepsilon) &= a_{11}x_1 + a_{12}x_2 + b_1(y - y^-(\varepsilon)) + \ldots \\
\bar{x}_2 - x_2^+(\varepsilon) &= a_{21}x_1 + a_{22}x_2 + b_2(y - y^-(\varepsilon)) + \ldots \\
\bar{y} &= y^+(\varepsilon) + c_1x_1 + c_2x_2 + d(y - y^-(\varepsilon))^2 + \ldots
\end{aligned}
$$

$$(2.6)$$

where $x_1^+(0) = x_1^+, x_2^+(0) = x_2^+, y^-(0) = y^-, y^+(0) = 0$; all coefficients $a_{11}, \ldots, d$ depend (smoothly) on ε; and we shift y^- into $y^-(\varepsilon)$ in order to nullify the linear in y terms from the right side of the third equation.

We assume that the following general condition holds

$$
b_1 c_1 b_2 c_2 \neq 0.
$$

$$(2.7)$$

Its meaning is as follows: if at least one of these four coefficients is zero, then in any neighborhood of f_0 there exist maps having a nontransversal homoclinic orbit close to Γ_0

6

with a non-simple quadratic tangency (see [2, 6, 4] for details); it composes an additional degeneracy which we do not consider here.

Conditions **A**, **B**, **C** together with (2.7) define a codimension 3 bifurcation surfaces of diffeomorphisms with a quadratic homoclinic tangency. Hence, as a general unfolding we should consider a three-parameter families where the parameters μ_1, μ_2 and μ_3 control the degeneracies imposed due to conditions **B**, **A** and **C**, respectively.

Naturally, the splitting distance of manifolds $W^s(O)$ and $W^u(O)$ with respect to the point M^+ is considered as the first governing parameter μ_1. It is seen from (2.6) that

$$\mu_1 \equiv y^+(\varepsilon) \ . \tag{2.8}$$

The second parameter should control the Jacobian $J = \lambda_1 \lambda_2 \gamma$ of f_μ at saddle O_μ. Therefore, we define

$$\mu_2 = 1 - |\lambda_1 \lambda_2 \gamma| \tag{2.9}$$

As the third parameter μ_3 we consider the value that controls the difference between $|\lambda_1|$ and $|\lambda_2|$, namely:

$$\mu_3 = \frac{|\lambda_1(\varepsilon)|}{|\lambda_2(\varepsilon)|} - 1 \tag{2.10}$$

Thus, the family f_{μ_1,μ_2,μ_3} constructed above can be considered as a general unfolding of the corresponding homoclinic tangency to a resonant saddle, satisfying conditions **A**, **B** and **C**.

Now we are able to construct the first return maps T_k using formulae (2.2) and (2.6). As a result we will obtain a formula for T_k in the initial (small) variables $(x_1, x_2, y) \in U_0$ and parameters μ_1, μ_2 and μ_3. Next, we rescale the initial variables and parameters

$$(x_1, x_2, y) \mapsto (X_1, X_2, Y) \ , \ (\mu_1, \mu_2, \mu_3) \mapsto (M_1, M_2, M_3) \ ,$$

with asymptotically small (as $k \to \infty$) factors, in such a way that in the rescaled variables and parameters map T_k is rewritten as some three-dimensional quadratic map which contains asymptotically small (as $k \to \infty$) terms. Moreover, new coordinates (X_1, X_2, Y) and parameters (M_1, M_2, M_3) can take arbitrary finite values at large k (i.e. covering all values in the limit $k \to \infty$).

Our main result is the following theorem.

Theorem 1 *Let f_{μ_1,μ_2,μ_3} be the family under consideration. Then, in the (μ_1, μ_2, μ_3)-parameter space, there exist infinitely many regions Δ_k accumulating at the origin as $k \to \infty$ such that the map T_k in appropriate rescaled coordinates and parameters is asymptotically C^{r-1}-close to the following limit map*

$$\bar{X}_1 \;=\; Y, \;\; \bar{X}_2 \;=\; X_1, \;\; \bar{Y} = M_1 + M_2 X_1 + BX_2 - Y^2 \;, \tag{2.11}$$

where

$$M_1 \;=\; -d\gamma^{2k}[\mu_1 + \lambda_1^k c_1 x_1^+ + \lambda_2^k c_2 x_2^+ + o(\lambda^k)] \tag{2.12}$$

and

$$M_2 = \left(b_1 c_1 + b_2 c_2 \frac{\lambda_2^k}{\lambda_1^k} \right) \lambda_1^k \gamma^k (1 + \ldots), \;\; B = J_1 (\lambda_1 \lambda_2 \gamma)^k (1 + \ldots) \tag{2.13}$$

3   Proof of Theorem 1.

Using (2.6) and (2.2) one can write the map $T_k = T_1 T_0^k$ for sufficiently large k and small ε in the form

$$\begin{aligned}
\bar{x}_1 - x_1^+ \;=\;& a_{11}(\lambda_1^k x_1 + \hat{\lambda}^k \xi_{k1}(x,y,\varepsilon)) + a_{12}(\lambda_2^k x_2 + \hat{\lambda}^k \xi_{k2}(x,y,\varepsilon)) + \\
&+ b_1 (y - y^-) + O(|y - y^-|^2 + \lambda^k \|x\| |y - y^-|) + \lambda^{2k} \|x\|^2) \;,
\end{aligned}$$

$$\begin{aligned}
\bar{x}_2 - x_2^+ \;=\;& a_{12}(\lambda_1^k x_1 + \hat{\lambda}^k \xi_{k1}(x,y,\varepsilon)) + a_{22}(\lambda_2^k x_2 + \hat{\lambda}^k \xi_{k2}(x,y,\varepsilon)) + \\
&+ b_2 (y - y^-) + O(|y - y^-|^2 + \lambda^k \|x\| |y - y^-|) + \lambda^{2k} \|x\|^2) \;,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
\gamma^{-k} \bar{y} - \hat{\gamma}^{-k} \eta_k(\bar{x}, \bar{y}, \varepsilon) \;=\;& \mu_1 + c_1 (\lambda_1^k x_1 + \hat{\lambda}^k \xi_{k1}(x,y,\varepsilon)) + \\
&+ c_2 (\lambda_2^k x_2 + \hat{\lambda}^k \xi_{k2}(x,y,\varepsilon)) + d(y - y^-)^2 + \\
&+ O(|y - y^-|^3 + \lambda^k \|x\| |y - y^-| + \lambda^{2k} \|x\|^2) \;.
\end{aligned}$$

We shift coordinates $x_{1new} \;=\; x_1 - x_1^+ + \phi_k^1(\varepsilon)$, $x_{2new} \;=\; x_2 - x_2^+ + \phi_k^2(\varepsilon)$, $y_{new} \;=\; y - y^- + \psi_k(\varepsilon)$, where $\phi_k, \psi_k = O(\lambda^k)$, in such a way that the right sides of (3.1) do not contain constant terms for the first two equations and linear in y_{new} terms for the third equation. Then (3.1) takes the form

$$\bar{x}_1 \;=\; a_{11}\lambda_1^k x_1 + a_{12}\lambda_2^k x_2 + b_1 y + O(y^2 + \lambda^k \|x\| |y| + \hat{\lambda}^k \|x\|^2) \;,$$

$$\bar{x}_2 \;=\; a_{21}\lambda_1^k x_1 + a_{22}\lambda_2^k x_2 + b_2 y + O(y^2 + \lambda^k \|x\| |y| + \hat{\lambda}^k \|x\|^2) \;, \tag{3.2}$$

$$\begin{aligned}
\bar{y} - (\hat{\gamma}/\gamma)^{-k} \eta_k(\bar{x} + x^+ + \phi_k, \bar{y} + y^- + \psi_k, \varepsilon) \;=\;& M_k + \\
d\gamma^k y^2 + \lambda_1^k \gamma^k c_1 x_1 + \lambda_2^k \gamma^k c_2 x_2 + \gamma^k O(|y|^3 + \lambda^k \|x\| |y|) &+ \hat{\lambda}^k \|x\|^2)
\end{aligned}$$

where

$$M_k \;=\; \gamma^k[\mu_1 + \lambda_1^k c_1 x_1^+ + \lambda_2^k c_2 x_2^+ + o(\lambda^k)] \tag{3.3}$$

8

Consider the third equation of (3.2). First of all, we transform its left side. Namely, we write $\bar{y} - (\hat{\gamma}/\gamma)^{-k}\eta_k = \bar{y} + (\hat{\gamma}/\gamma)^{-k}[\eta_k^0 + \eta_k^1(\bar{x}, \varepsilon) + \eta_k^2(\bar{y}, \varepsilon) + \eta_k^3(\bar{x}, \bar{y}, \varepsilon)]$ where $\eta_k^1(0, \varepsilon) = 0$, $\eta_k^2(0, \varepsilon) = 0$ and $\eta_k^3 = O(\|\bar{x}\bar{y}\|)$. Next, we transfer constant term $(\hat{\gamma}/\gamma)^{-k}\eta_k^0$ into the right side and join it to M_k^1; we substitute the value of $\bar{x}$ due to the first two equations of (3.2) into function $\eta_k^1(\bar{x}, \varepsilon)$ and transfer the obtained expression into the right side. After this, all coefficients (in the third equation) get additions of order $O(\hat{\gamma}^{-k})$ and a new linear term in y, $p_k y = O([\hat{\gamma}/\gamma]^{-k})y$, appears. By the shift of coordinates of the form $(x, y) \mapsto (x, y) + O([\hat{\gamma}/\gamma]^{-k})$, we vanish both this linear term and constant terms in the right sides of the first and second equations. As the result, the left side of the third equation can be written as follows:

$$\bar{y} + (\hat{\gamma}/\gamma)^{-k}O(\bar{y}) + (\hat{\gamma}/\gamma)^{-k}O(\|\bar{x}\bar{y}\|) = \bar{y}(1 + q_k) + (\hat{\gamma}/\gamma)^{-k}O(\bar{y}^2) + (\hat{\gamma}/\gamma)^{-k}O(\|\bar{x}\bar{y}\|)$$

where $q_k = O([\hat{\gamma}/\gamma]^{-k})$. After this, we can write system (3.2) in the form

$$\begin{aligned}
\bar{x}_1 &= a_{11}\lambda_1^k x_1 + a_{12}\lambda_2^k x_2 + b_1 y + O(y^2) + \lambda^k O(\|x\|\|y\|) + \hat{\lambda}^k O(\|x\|^2) \,, \\
\bar{x}_2 &= a_{21}\lambda_1^k x_1 + a_{22}\lambda_2^k x_2 + b_2 y + O(y^2) + \lambda^k O(\|x\|\|y\|) + \hat{\lambda}^k O(\|x\|^2) \,,
\end{aligned}$$

$$\begin{aligned}
\bar{y}(1 + q_k) + (\hat{\gamma}/\gamma)^{-k}O(|\bar{y}|^2) + (\hat{\gamma}/\gamma)^{-k}O(\|\bar{x}\bar{y}\|) &= M_k + \\
+ d\gamma^k(1 + s_k)y^2 + c_1\lambda_1^k\gamma^k x_1 + c_2\lambda_2^k\gamma^k x_2 + p_k\gamma^k O(\|x\|^2) + \\
+ \lambda^k\gamma^k O(\|x\|\|y\|) + \gamma^k O(y^3) \,,
\end{aligned}$$

(3.4)

where $s_k = O(\lambda^k + |\hat{\gamma}/\gamma|^{-k})$, $p_k = O(\hat{\lambda}^k + |\hat{\gamma}/\gamma|^{-k})$ and new M_k satisfies (3.3).

We perform a linear change of x variables to make zero the linear in y term in the second equation:

$$x_{2new} = x_2 - \frac{b_2}{b_1}x_1, \ x_{1new} = x_1 \,, \ y_{new} = y.$$

Then (3.4) is rewritten in the form

$$\bar{x}_1 = b_1 y + \lambda^k O(\|x\|) + O(y^2) \,,$$

$$\bar{x}_2 = A_{21}\lambda_1^k x_1 + A_{22}\lambda_2^k x_2 + O(y^2) + \lambda^k O(\|x\|\|y\|) + \lambda^{2k}O(\|x\|^2) \,,$$

$$\begin{aligned}
\bar{y}(1 + q_k) + (\hat{\gamma}/\gamma)^{-k}O(|\bar{y}|^2 + \|\bar{x}\bar{y}\|) &= M_k + \lambda_1^k\gamma^k c_1\nu_k x_1 + \lambda_2^k\gamma^k c_2 x_2 + d\gamma^k(1 + s_k)y^2 + \\
p_k\gamma^k O(\|x\|^2) + \lambda^k\gamma^k O(\|x\|\|y\|) + \gamma^k O(y^3) \,,
\end{aligned}$$

(3.5)

where

$$A_{21} = \left(a_{21} - \frac{b_2}{b_1}a_{11}\right) + \left(a_{22} - \frac{b_2}{b_1}a_{12}\right)\frac{b_2}{b_1}\frac{\lambda_2^k}{\lambda_1^k},$$

$$A_{22} = a_{22} - \frac{b_2}{b_1}a_{12}, \qquad (3.6)$$

$$\nu_k = \left(1 + \frac{b_2 c_2}{b_1 c_1}\frac{\lambda_2^k}{\lambda_1^k}\right).$$

Now we will vary λ_1 and λ_2 in such a way that the value of ν_k is asymptotically small as $k \to \infty$. This is always possible via small changes of parameter μ_3 because $b_1 c_1 \neq 0$ and $b_2 c_2 \neq 0$ and $\lambda_1 = -\lambda_2$ in the initial moment. Then it is clear that

$$A_{21} = \frac{J_1}{c_2 b_1} + O(\nu_k) \neq 0,$$

where J_1 is given by formula (2.5).

Rescale the coordinates as follows

$$y = -\frac{\gamma^{-k}(1 + q_k)}{d(1 + s_k)}\,Y \; , \; x_1 = -\frac{b_1 \gamma^{-k}(1 + q_k)}{d(1 + s_k)}\,X_1 \; , \; x_2 = -\frac{b_1 A_{21}\gamma^{-k}\lambda_1^k(1 + q_k)}{d(1 + s_k)}\,X_2.$$

Then system (3.5) is rewritten in the new coordinates as follows

$$\begin{aligned}
\bar{X}_1 &= Y + O(\lambda^k)\,, \\
\bar{X}_2 &= X_1 + O(\gamma^{-k}\lambda^{-k})\,, \\
\bar{Y} &= M_1 + M_2 X_1 + B X_2 - Y^2 + O(\gamma^{-k}\lambda^{-k})\,,
\end{aligned} \qquad (3.7)$$

where formulas (2.12) and (2.13) are valid for M_1, M_2 and B.

It is obvious that system (3.7) is asymptotically close to (2.11) when $k \to \infty$. $\quad\square$.

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